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Mathematical modeling of magneto-sensitive elastomers

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Abstract

Magneto-sensitive (MS) elastomers are a class of smart materials whose mechanical properties change instantly by the application of a magnetic field. These materials typically consist of micron-sized ferrous particles dispersed in an elastomer. The full system of equations for deformable MS solids in an electro-magnetic field is first considered. Then, the strain-energy functions for isotropic MS elastomers are presented and a simple phenomenological model is suggested. Finally, to illustrate some of the features of the derived model, a MS elastomer confined by parallel top and bottom plates is subjected to shear deformation under the influence of a magnetic field normal to the plates. An acceptable agreement is illustrated between numerical simulation and experimental observation.

Keywords: Magnetoelasticity; Composite elastomer; Finite deformation; Constitutive laws; Mathematical modeling

1. Introduction

Magneto-sensitive (MS) (in literature magnetorheological—MR) elastomers are materials that respond to an applied magnetic field with an instantaneous change in the mechanical behavior. An improved understanding of MS elastomers is demanded by the prospect to provide simple, reliable and rapid-response interfaces between controls laws and mechanical systems. It is now well recognized that MS elastomers have the potential to improve the design of electromechanical devices and their operation. For example, an elastomer with field dependent properties may be used as a device with a variable stiffness. Therefore, this wide range of potential applications and associated economic benefits are the reason for the intense research on these materials in recent years, see for example Borcea and Bruno (2001), Carlson and Jolly (2000) and Jolly et al. (1996a).

The magnetic efficiency to change field dependent mechanical properties is optimized by choosing a particle material with a high magnetic saturation. Cobalt has the largest saturation of 2.4 T of all known elements, however it is not used commercially. In general an alloy of iron with a magnetic saturation of 2.1 T is used as an additive in the mixing process of MS elastomeric compounds.

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MS solid elastomers are less well publicized and recognized than their electro-sensitive (ES) counterparts. Both elastomers are composed of polarizable particles, dispersed in a polymer medium, having a size on the order of a few microns (typically from 10^{-7} to 10^{-5} m). Carrier fillers are selected based upon their electro-magnetic and thermo-mechanical properties: silicone and/or other rubber-like materials with very small electric conductivity. Typical particle volume fractions are between 0.1 and 0.5. During the manufacturing process of MS elastomers, the isotropy condition inherent of the filler material is maintained in the final composite. Therefore, these materials are considered isotropic and non-conductive. We assume, for simplicity, that they remain isotropic even during the application of a magnetic field. However, MS elastomers become non-homogeneous due to the presence and distribution of particles in the carrier filler.

Structurally, field responsive elastomers can be thought of as the solid analogs of field responsive non-colloidal suspensions or fluids, see for example Ginder and Davis (1994). There are however some distinct differences in the way in which these two classes of materials are intended to be used. The most important one is that the field sensitive particles within the elastomer composite are intended to always operate in the pre-yield regime, see Carlson and Jolly (2000). Therefore, field responsive elastomers can best be described by a field dependent modulus, see for example Borcea and Bruno (2001), Jolly et al. (1996a), Rigbi and Jilken (1983).

Jacob Rabinow at the US National Bureau of Standards introduced MS materials in the late 1940s, see Rabinow (1948). At about the same time, initial experimental results on electrorheological fluids were published by Winslow (1949). Following the discovery of these materials, the major interest was dedicated towards ES materials in the late 1940s and early 1950s, which is evident by the larger number of patents and publications. Except for the interest immediately following Rabinow's work, there has been little new information about and publications on MS media. Recently however, MS elastomers have been recognized as a commercially viable product and thus, an increase in research and publication is recognized (see for example Kordonsky, 1993; Carlson and Jolly, 2000; Jolly et al., 1996a). A number of MS elastomers and various MS elastomer-based systems have successfully been brought into the market, including adaptive tuned vibration absorbers, stiffness tunable mounts, suspensions and automotive bushing.

In the following analysis, we consider the MS elastomer as a moving non-polar isotropic continuum. Section 2 starts with the classical work by Pao (1978). We summarize the full system of equations for the moving isotropic non-polar continuum medium in an electro-magnetic field such as the Maxwell equations, the mechanical and thermodynamical balance laws. In Section 3 we derive the basic system of constitutive equations for MS elastomers using a phenomenological approach based on experimental data by Carlson and Jolly (2000). The reduced system of constitutive equations is complemented by the system of boundary and initial conditions. In Section 4 we present and analyze a simple strain-energy function for MS elastomers. For illustration of the presented phenomenological model, in Section 5 we examine the basic operational features of controllable incompressible elastomeric devices. The final section is devoted to concluding remarks.

2. Physical laws for moving continuum media in an electro-magnetic field

Let a continuum deformable solid in the reference configuration occupy a domain $\Omega \subset \mathbb{R}^3$. In the deformed configuration each point $\mathbf{X} \in \overline{\Omega}$ moves into the position $\mathbf{x} = \mathbf{X} + \mathbf{u} = \boldsymbol{\chi}(t, \mathbf{X}) \in \mathbb{R}^3$ where $\mathbf{u}$ and $\boldsymbol{\chi}$ are the displacement and mapping, respectively, and t is the parameter describing the motion of the medium (usually, this is the physical time). We consider a one-to-one, i.e. locally invertible and orientation-preserving mapping in $\overline{\Omega}$ for every $t > 0$.

In this paper standard mathematical notations are used; for more detail we refer to the textbook by Lurie (1990). Namely, for the scalar φ, vectors $\mathbf{a}$, $\mathbf{b}$ and second-order tensors $\mathbf{A}$, $\mathbf{B}$ the algebraic rules are

$$\mathbf{a}\cdot\mathbf{b}=a_ib_i,\quad |\mathbf{a}|^2=\mathbf{a}\cdot\mathbf{a},$$
$$\mathbf{a}\times\mathbf{b}=\{\varepsilon_{ijk}a_ib_j\}_{k=1,2,3},\quad \mathbf{ab}=\{a_ib_j\}_{i,j=1,2,3},$$
$$\mathbf{A}:\mathbf{B}=A_{ij}B_{ij},\quad |\mathbf{A}|^2=\mathbf{A}:\mathbf{A},\quad \mathbf{A}\cdot\mathbf{b}=\{A_{ij}b_j\}_{i=1,2,3},$$

and the spatial differential operators of divergence, gradient and rotor are given by

$$\operatorname{div}\mathbf{a}=\frac{\partial a_i}{\partial x_i},\quad \operatorname{div}\mathbf{A}=\left\{\frac{\partial A_{ij}}{\partial x_j}\right\}_{i=1,2,3},\quad \partial_{\mathbf{x}}\varphi=\frac{\partial\varphi}{\partial\mathbf{x}}=\left\{\frac{\partial\varphi}{\partial x_i}\right\}_{i=1,2,3},$$
$$\partial_{\mathbf{x}}\mathbf{a}=\frac{\partial\mathbf{a}}{\partial\mathbf{x}}=\left\{\frac{\partial a_i}{\partial x_j}\right\}_{i,j=1,2,3},\quad \operatorname{rot}\mathbf{a}=\left\{\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\right\}_{k=1,2,3},$$

where over repeated indices the summation rule applies. For the basis vectors $\mathbf{e}_i$ the permutation or the Levi–Civita symbol is $\varepsilon_{ijk}=(\mathbf{e}_i\times\mathbf{e}_j)\cdot\mathbf{e}_k$ with $\varepsilon_{ijk}=1$ or -1 according to whether the indices are in a cyclic or an anticyclic order, respectively, and $\varepsilon_{ijk}=0$ otherwise.

The gradient of mapping or deformation tensor $\mathbf{F}$, the velocity $\mathbf{v}$, its spatial gradient $\mathbf{l}$ and its deformation tensor $\mathbf{d}$ are respectively defined by

$$\mathbf{F}=\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}},\quad \mathbf{v}=\frac{\partial\boldsymbol{\chi}}{\partial t},\quad \mathbf{l}=\partial_{\mathbf{x}}\mathbf{v},\quad \mathbf{d}=\frac{1}{2}(\mathbf{l}+\mathbf{l}^{\mathrm{T}}),$$

where the superscript T denotes the transpose of a tensor.

For the total time derivative of a scalar-valued function $\varphi=\varphi(t,\mathbf{x})$ and a vector-valued function $\mathbf{a}=\mathbf{a}(t,\mathbf{x})$ the classic formulas of Euler are used, respectively,

$$\dot{\varphi}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\frac{\partial\varphi}{\partial t}+\mathbf{v}\cdot\partial_{\mathbf{x}}\varphi,$$
$$\dot{\mathbf{a}}=\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}=\frac{\partial\mathbf{a}}{\partial t}+\mathbf{v}\cdot\partial_{\mathbf{x}}\mathbf{a}.$$

In the most general case the moving non-polar deformable solid in the present (current or actual) configuration is described by

- *electro-magnetic variables* such as the electric field intensity **E**, the magnetic field intensity **H**, the electric induction or displacement **D**, the electric polarization density **P** (electric moment per unit volume), the magnetic flux density or magnetic induction **B**, the magnetic polarization (magnetization) density or intrinsic induction **M** (magnetic moment per unit volume), the free electric current density **J** and the free electric charge density q (see e.g., Eringen and Maugin, 1989; Jackson, 1983; Pao, 1978);
- *mechanical variables* such as the deformation tensor **F**, the velocity **v** and its spatial gradient **l**, the density ρ, the stress tensor $\boldsymbol{\sigma}$ and the external mechanical body force density **f** per unit mass. For more detail we refer to the textbooks by Holzapfel (2001), Lurie (1990) and Truesdell (1991);
- *thermodynamic variables* such as the absolute temperature T and the Cauchy heat flux **Q**, (see e.g., Holzapfel, 2001; Müller and Rugerri, 1993).

For these variables the appropriate physical laws are well known.

The Maxwell equations for continuum media are presented here in the standard meter–kilogram–second–Coulomb units, as discussed in detail by Eringen and Maugin (1989), Jackson (1983) and Pao (1978).

The electric field intensity **E** and the magnetic flux density **B** are regarded as the basic variables in a vacuum. For condensed media additional variables, such as the electric polarization density **P** and the

magnetic polarization density $\mathbf{M}$ are introduced. The interconnection of these variables are shown in the following relations [1]

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \tag{2.1}$$

where $\varepsilon_0 \approx 8.85 \times 10^{-12}$ and $\mu_0 = 4\pi \times 10^{-7} \approx 1.26 \times 10^{-6}$ are two universal constants (the second one is the magnetic permeability of a vacuum). These constants are related by the equation $\varepsilon_0 \mu_0 = c^{-2}$ where c is the speed of light in vacuum.

The Gauss, Faraday and Ampere laws are respectively given by

$$\varepsilon_0 \operatorname{div}(\mathbf{E}) = q - \operatorname{div} \mathbf{P}, \tag{2.2}$$

$$\operatorname{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{2.3}$$

$$\mu_0^{-1} \operatorname{rot} \mathbf{B} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mu_0^{-1} \operatorname{rot} \mathbf{M} + \mathbf{J}, \tag{2.4}$$

where $-\operatorname{div} \mathbf{P}$ is the polarization charge, $\partial \mathbf{P}/\partial t$ is the polarization current, and $\mu_0^{-1} \operatorname{rot} \mathbf{M}$ is the magnetization current.

The conservation of magnetic flux density reads

$$\operatorname{div} \mathbf{B} = 0. \tag{2.5}$$

The equation for the conservation of electric charge $\partial q/\partial t + \operatorname{div} \mathbf{J} = 0$ is not an independent equation, which follows from Eqs. (2.2) and (2.4) and the identity $\operatorname{div} \operatorname{rot}(*) \equiv 0$.

Mechanical balance laws for solids in the local form are the conservation of mass for closed systems and the balance of linear momentum in the following forms, respectively,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \tag{2.6}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} + \mathbf{f}_{\mathrm{e}}, \tag{2.7}$$

where $\mathbf{f}_{\mathrm{e}}$ is the electro-magnetic force. From the Dipole-Current Circuit Model for moving continuum medium this force is given by Pao (1978)

$$\mathbf{f}_{\mathrm{e}} = q\mathbf{E} + \mathbf{J} \times \mathbf{B} + \mu_0^{-1} (\partial_{\mathbf{x}} \mathbf{B}) \cdot \mathbf{M} + (\partial_{\mathbf{x}} \mathbf{E}) \cdot \mathbf{P} + \frac{\partial}{\partial t} (\mathbf{P} \times \mathbf{B}) + \operatorname{div}(\mathbf{v}\mathbf{P} \times \mathbf{B}). \tag{2.8}$$

It is further shown by Pao (1978) that the balance of angular momentum takes the local form

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + (\mu_0^{-1} \mathbf{M} + \mathbf{v} \times \mathbf{P}) \times \mathbf{B} + \mathbf{P} \times \mathbf{E} = \mathbf{0}, \tag{2.9}$$

where $\boldsymbol{\varepsilon}$ denotes the third-order permutation tensor with the Levi–Civita symbol as its components.

The laws of thermodynamics have the following form (see e.g., Holzapfel, 2001; Müller and Rugerri, 1993; Pao, 1978).

The first law is the balance of energy in the local form

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} \left(U + \frac{1}{2} |\mathbf{v}|^2 \right) + \operatorname{div} \mathbf{Q} = \operatorname{div}(\boldsymbol{\sigma} \cdot \mathbf{v}) + \rho \mathbf{f} \cdot \mathbf{v} + \rho R + w_{\mathrm{e}}, \tag{2.10}$$

where U and R denote the specific, i.e. per unit mass, internal energy and the radiant heating, respectively. Here w_{e} is the electro-magnetic power, which is defined through

[1] Note, that the relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ has also been used by Pao (1978). Here we replace $\mathbf{M}$ by $\mu_0^{-1}\mathbf{M}$ in all relations.

$$w_e = \mathbf{f}_e \cdot \mathbf{v} + \mathbf{J}_e \cdot \mathbf{E}_e - \mathbf{M}_e \cdot \frac{d\mathbf{B}}{dt} + \rho \frac{d}{dt}\left(\frac{\mathbf{P}}{\rho}\right) \cdot \mathbf{E}_e, \tag{2.11}$$

where $\mathbf{J}_e$ is the effective conduction current, $\mathbf{E}_e$ is the effective electric field intensity and $\mathbf{M}_e$ is the effective magnetization in the rest frame, which according to Minkowski theory (1908) are related to the laboratory frame variables by

$$\mathbf{J}_e = \mathbf{J} - q\mathbf{v}, \quad \mathbf{E}_e = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad \mathbf{M}_e = \mu_0^{-1}\mathbf{M} + \mathbf{v} \times \mathbf{P}. \tag{2.12}$$

Using Eqs. (2.6) and (2.7) we easily get the reduced form of Eq. (2.10), which reads

$$\rho\dot{U} + \operatorname{div}\mathbf{Q} = \boldsymbol{\sigma} : \mathbf{l} - \mathbf{M}_e \cdot \dot{\mathbf{B}} + \mathbf{J}_e \cdot \mathbf{E}_e + \rho R + \dot{\mathbf{P}} \cdot \mathbf{E}_e + (\mathbf{P} \cdot \mathbf{E}_e)\operatorname{div}\mathbf{v}. \tag{2.13}$$

From a great number of formulations (not equivalent) of the second law of thermodynamics, we select the local formulation of the Clausius–Duhem inequality

$$\rho\frac{dS}{dt} + \operatorname{div}\left(\frac{\mathbf{Q}}{T}\right) - \rho\frac{R}{T} \geqslant 0, \tag{2.14}$$

where S is the specific entropy. The law describes the requirement that the internal entropy does not decrease in time. For more detail we refer to the textbooks by Holzapfel (2001), Lurie (1990) and Truesdell (1991).

Introducing the well known specific Helmholtz free energy Ψ through

$$\Psi = U - TS - \frac{1}{\rho}\mathbf{E}_e \cdot \mathbf{P}, \tag{2.15}$$

using the standard relation

$$\boldsymbol{\sigma} : \mathbf{l} = (\mathbf{F}^{-1} \cdot \boldsymbol{\sigma})^{\mathrm{T}} : \dot{\mathbf{F}}, \tag{2.16}$$

and substituting Eqs. (2.15) and (2.10) into Eq. (2.14) we obtain the main dissipation inequality

$$-\rho(\dot{\Psi} + \dot{T}S) + (\mathbf{F}^{-1} \cdot \boldsymbol{\sigma})^{\mathrm{T}} : \dot{\mathbf{F}} - \mathbf{M}_e \cdot \dot{\mathbf{B}} - \frac{1}{T}\mathbf{Q} \cdot \partial_{\mathbf{x}} T + \mathbf{J}_e \cdot \mathbf{E}_e - \mathbf{P} \cdot \dot{\mathbf{E}}_e \geqslant 0. \tag{2.17}$$

We shall name the system of equations (2.2)–(2.7), (2.9), (2.13) and the inequality (2.17) as the *basic system*.

3. Reduction of the basic system for hyperelastic MS solids

It can easily be verified that the basic system is indeterminant, i.e. there are more unknown variables than equations. The system is rendered determinate by providing a sufficient number of constitutive material laws. Here we consider only hyperelastic solids, thus no viscosity dependence needs to be accounted for. For these solids the stress tensor and the work done by the stresses do not depend on the path of deformation, which takes the solid from the reference to the current configuration. Hyperelasticity implies a conservative structure, i.e. the stress is obtained as the derivative of the scalar Helmholtz free-energy function with respect to a work conjugate strain field (see for example Holzapfel, 2001; Lurie, 1990; Ogden, 1997; Truesdell, 1991).

From a thermodynamical point of view, the variables $(\mathbf{v}, \mathbf{F}, \mathbf{E}_e, \mathbf{B}, T, \partial_{\mathbf{x}} T)$ are independent quantities for hyperelastic materials as discussed by Truesdell (1991) and Truesdell and Noll (1992) and thus we need to define constitutive relations for $(U, S, \boldsymbol{\sigma}, \mathbf{P}, \mathbf{J}_e, \mathbf{M}_e, \mathbf{Q})$.

The heat flux vector $\mathbf{Q}$ is given by the Fourier law of heat conduction as shown for example by Müller and Rugerri (1993)

$$\mathbf{Q} = -k\partial_{\mathbf{x}} T, \tag{3.1}$$

where $k > 0$ is the thermal conductivity, which can be assumed constant for almost every engineering material. For all remaining constitutive relations we drop the dependence on the temperature gradient $\partial_{\mathbf{x}} T$.

From the principle of material objectivity as shown by Lurie (1990) and Truesdell (1991), for example, it follows that all constitutive relations are independent on $\mathbf{v}$. As a result, we have

$$\mathbf{\Phi} = \hat{\mathbf{\Phi}}(T, \mathbf{F}, \mathbf{E}_{\mathrm{e}}, \mathbf{B}), \tag{3.2}$$

where $\mathbf{\Phi} = (U, S, \boldsymbol{\sigma}, \mathbf{P}, \mathbf{J}_{\mathrm{e}}, \mathbf{M}_{\mathrm{e}})$ is the generalized vector of unknown quantities.

Using Eqs. (3.1) and (3.2) and computing $\dot{\Psi}$ in Eq. (2.17) we obtain the following form of the Clausius–Duhem inequality:

$$-\rho\left(\frac{\partial \Psi}{\partial T} + S\right)\dot{T} - \left(\rho\frac{\partial \Psi}{\partial \mathbf{F}} - (\mathbf{F}^{-1}\cdot\boldsymbol{\sigma})^{\mathrm{T}}\right) : \dot{\mathbf{F}} - \left(\rho\frac{\partial \Psi}{\partial \mathbf{E}_{\mathrm{e}}} + \mathbf{P}\right)\cdot\dot{\mathbf{E}}_{\mathrm{e}} - \left(\rho\frac{\partial \Psi}{\partial \mathbf{B}} + \mathbf{M}_{\mathrm{e}}\right)\cdot\dot{\mathbf{B}} + k\frac{|\partial_{\mathbf{x}} T|^2}{T} + \mathbf{J}_{\mathrm{e}}\cdot\mathbf{E}_{\mathrm{e}} \geqslant 0. \tag{3.3}$$

This inequality must be satisfied at all times and at every fixed point in space for all *admissible thermodynamic processes*, i.e. processes compatible with the balance laws and the constitutive response functions. Since the quantities $\dot{T}$, $\dot{\mathbf{F}}$, $\dot{\mathbf{E}}_{\mathrm{e}}$ and $\dot{\mathbf{B}}$ are independent, see for example Truesdell and Noll (1992), and since the inequality in Eq. (3.3) is linear in these rates, we obtain

$$S = -\frac{\partial \Psi}{\partial T}, \quad (\mathbf{F}^{-1}\cdot\boldsymbol{\sigma})^{\mathrm{T}} = \rho\frac{\partial \Psi}{\partial \mathbf{F}}, \quad \mathbf{P} = -\rho\frac{\partial \Psi}{\partial \mathbf{E}_{\mathrm{e}}}, \quad \mathbf{M}_{\mathrm{e}} = -\rho\frac{\partial \Psi}{\partial \mathbf{B}}, \tag{3.4}$$

and the reduced dissipation inequality is given by

$$k\frac{|\partial_{\mathbf{x}} T|^2}{T} + \mathbf{J}_{\mathrm{e}}\cdot\mathbf{E}_{\mathrm{e}} \geqslant 0. \tag{3.5}$$

Isotropic continuum media is described by the following well-known experimental laws involving the electro-magnetic variables previously introduced in Eq. (2.1), see Jackson (1983) and Pao (1978)

$$\mathbf{J} = \eta\mathbf{E}, \quad \mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}, \quad \mathbf{B} = \mu\mu_0\mathbf{H}, \tag{3.6}$$

where the scalar functions $\eta = \eta(\mathbf{x}, \mathbf{E}) \geqslant 0$, $\varepsilon = \varepsilon(\mathbf{x}, \mathbf{E}) \geqslant 1$ and $\mu = \mu(\mathbf{x}, \mathbf{H}) \geqslant 1$ are the electric conductivity, the relative dielectric permittivity and the relative magnetic permeability, respectively. In the vacuum $\eta \equiv 0$, $\varepsilon \equiv 1$ and $\mu \equiv 1$.

From Eqs. (2.1), (2.12) and (3.6) we obtain the constitutive relations for the effective conduction current $\mathbf{J}_{\mathrm{e}}$ and the effective magnetization $\mathbf{M}_{\mathrm{e}}$ as

$$\mathbf{J}_{\mathrm{e}} = \eta\mathbf{E} - q\mathbf{v}, \quad \mathbf{M}_{\mathrm{e}} = \gamma\mathbf{B} + (\varepsilon - 1)\varepsilon_0\mathbf{v}\times\mathbf{E}, \tag{3.7}$$

where $\gamma = (\mu - 1)(\mu_0\mu)^{-1} \geqslant 0$ is the function of magnetic saturation, which is easily defined from the standard $\mu_0|H| \mapsto |B|$ experimental curve, see for example Jolly et al. (1996a).

In this and the following sections some of our assumptions are based on experimental data as given by Carlson and Jolly (2000) and Jolly et al. (1996a).

The main assumption is that MS materials have no electric polarization

$$\mathbf{P} = \mathbf{0}, \tag{3.8}$$

i.e. $\varepsilon \equiv 1$ in all relations. From Eq. $(3.7)_2$ we have $\mathbf{M}_{\mathrm{e}} = \mu_0^{-1}\mathbf{M} = \gamma\mathbf{B}$. As a result, from the law of the balance of angular momentum in the form shown by Eq. (2.9), it follows that the stress tensor $\boldsymbol{\sigma}$ is symmetric, i.e. it is the *Cauchy stress tensor* and thus the equality $\boldsymbol{\sigma}:\mathbf{l} = \boldsymbol{\sigma}:\mathbf{d}$ applies.

It follows from the above assumptions and relations that the basic system can now be rewritten as

$$\operatorname{div}\mathbf{E} = \frac{q}{\varepsilon_0}, \tag{3.9}$$

$$\operatorname{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{3.10}$$

$$\operatorname{rot}(\mu^{-1}\mathbf{B}) = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\mathrm{d}t} + \mu_0\eta\mathbf{E}, \tag{3.11}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{3.12}$$

$$\dot{\rho} + \rho\operatorname{div}\mathbf{v} = 0, \tag{3.13}$$

$$\rho\dot{\mathbf{v}} = \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{f} + \mathbf{f}_{\mathrm{e}}, \tag{3.14}$$

$$\rho\dot{U} - k\Delta T = \boldsymbol{\sigma} : \mathbf{d} - \gamma\mathbf{B}\cdot\dot{\mathbf{B}} + \rho R + \mathbf{E}\cdot\mathbf{G}, \tag{3.15}$$

$$k\frac{|\partial_{\mathbf{x}} T|^2}{T} + \mathbf{E}\cdot\mathbf{G} \geqslant 0, \tag{3.16}$$

where $\mathbf{f}_{\mathrm{e}}$ and $\mathbf{G}$ are given, respectively, by

$$\mathbf{f}_{\mathrm{e}} = q\mathbf{E} + \eta\mathbf{E}\times\mathbf{B} + \gamma(\partial_{\mathbf{x}}\mathbf{B})\cdot\mathbf{B},$$

$$\mathbf{G} = \eta(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - q\mathbf{v}. \tag{3.17}$$

In the system of constitutive equations (3.9)–(3.16) some variables and expressions differ by several orders of magnitude in problems of practical interest. Of course, we could repeat the non-dimensional analysis of constitutive relations used by Rajagopal and Růžička (2001), but we prefer evaluating experimental data of commercial MS elastomers given for example by Carlson and Jolly (2000) and Jolly et al. (1996a).

We mentioned in the introduction that for commercially available MS elastomers, the electric charge and the electric conductivity are very small. Therefore, we shall assume that $q \equiv 0$ and $\eta \equiv 0$, and from Eqs. $(3.7)_1$ and (3.17) it follows that $\mathbf{J}_{\mathrm{e}} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$.

Secondly, experimental data for MS elastomers show that the influence of the electric filed is non-essential, and therefore, we rewrite Eq. (3.2) as

$$\boldsymbol{\Phi} = \hat{\boldsymbol{\Phi}}(T, \mathbf{F}, \mathbf{B}), \tag{3.18}$$

where the generalized vector of unknown values for MS elastomers now becomes $\boldsymbol{\Phi} = (U, \boldsymbol{\sigma})$.

From Eqs. $(3.4)_2$, $(3.4)_3$ and (3.8) it follows that the free energy function Ψ is independent in $\mathbf{E}_{\mathrm{e}}$. Moreover, it is assumed that the Helmholtz free energy $\Psi = \Psi(T, \mathbf{F}, \mathbf{B})$ is a smooth function, see for example Holzapfel (2001) and Truesdell (1991). Then, using Eq. (2.15) with $\mathbf{P} = \mathbf{0}$, Eqs. (3.4) and (3.7) with $\mathbf{M}_{\mathrm{e}} = \gamma\mathbf{B}$ and Eq. (2.16), we obtain the following relation for the rate of the internal energy:

$$\begin{aligned}\rho\dot{U} &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi - T\frac{\partial\Psi}{\partial T}\right) = \rho\frac{\partial\Psi}{\partial\mathbf{F}} : \dot{\mathbf{F}} + \rho\frac{\partial\Psi}{\partial\mathbf{B}}\cdot\dot{\mathbf{B}} - \rho T\frac{\partial}{\partial T}\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right)\\ &= \left(\boldsymbol{\sigma} - T\frac{\partial\boldsymbol{\sigma}}{\partial T}\right) : \mathbf{d} - \gamma\mathbf{B}\cdot\dot{\mathbf{B}} + \rho\left(-T\frac{\partial^2\Psi}{\partial T^2}\right)\dot{T} - \rho T\frac{\partial^2\Psi}{\partial T\partial\mathbf{B}}\cdot\dot{\mathbf{B}}.\end{aligned} \tag{3.19}$$

The *specific heat capacity* is defined, for example in Müller and Rugerri (1993), or Truesdell (1991), as

$$c_{\mathrm{v}} := -T\frac{\partial^2\Psi}{\partial T^2} > 0, \tag{3.20}$$

and using Eq. (3.19), we rewrite Eq. (3.15) in the form

$$\rho c_{\mathrm{v}}\dot{T} - k\Delta T - \rho R = T\frac{\partial \boldsymbol{\sigma}}{\partial T} : \mathbf{d} + \rho T\frac{\partial^2 \Psi}{\partial T \partial \mathbf{B}} \cdot \dot{\mathbf{B}}.$$

As it follows from Eq. (2.6) and from the independence of the thermodynamic variables $\mathbf{F}$ and T, the density ρ as well as the magnetic field intensity $\mathbf{H}$ do not depend on the temperature T, therefore, from Eqs. (3.4), (3.6) and (3.7) we have

$$\rho T\frac{\partial^2 \Psi}{\partial T \partial \mathbf{B}} = -T\frac{\partial}{\partial T}(\gamma \mathbf{B}) = -T\frac{\partial(\mu - 1)}{\partial T}\mathbf{H} = -\frac{T}{\mu_0 \mu}\frac{\partial \mu}{\partial T}\mathbf{B}.$$

As a result, the final reduction of the basic system gives

$$\operatorname{div} \mathbf{E} = 0, \tag{3.21}$$

$$\operatorname{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{3.22}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{3.23}$$

$$\operatorname{rot}(\mu^{-1}\mathbf{B}) = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\mathrm{d}t}, \tag{3.24}$$

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \tag{3.25}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} + \gamma(\partial_{\mathbf{x}}\mathbf{B}) \cdot \mathbf{B}, \tag{3.26}$$

$$\rho c_{\mathrm{v}}\dot{T} - k\Delta T - \rho R = T\frac{\partial \boldsymbol{\sigma}}{\partial T} : \mathbf{d} - \varphi \mathbf{B} \cdot \dot{\mathbf{B}}, \tag{3.27}$$

where the material functions

$$\gamma = \frac{\mu - 1}{\mu_0 \mu}, \quad \varphi = \frac{T}{\mu_0 \mu}\frac{\partial \mu}{\partial T} \tag{3.28}$$

are defined from experimental constitutive curves $\mu_0|H| \mapsto |B|$ at different temperatures. The main thermodynamic inequality given by Eq. (3.16) is fulfilled automatically because the thermal conductivity k and absolute temperature T are positive.

Every set of fields $(\mathbf{E}, \mathbf{B}, \mathbf{v}, \boldsymbol{\sigma}, T)$ satisfying the system of constitutive equations (3.21)–(3.27) is named *admissible*. Assuming that the inside and surface electric charge and the electric current are absent, the *solution* of the system (3.21)–(3.27) is the set of admissible fields satisfying the following boundary conditions:

(i) for electro-magnetic fields $\mathbf{E}$ and $\mathbf{B}$

$$\mathbf{n} \cdot [\mathbf{E}] = 0, \quad \mathbf{n} \cdot [\mathbf{B}] = 0, \quad \mathbf{n} \times [\mathbf{E} + \mathbf{v} \times \mathbf{B}] = \mathbf{0}, \quad \mathbf{n} \times [\mu^{-1}\mathbf{B} - c^{-2}\mathbf{v} \times \mathbf{E}] = \mathbf{0}, \tag{3.29}$$

(ii) for the velocity $\mathbf{v}$, the Cauchy stress tensor $\boldsymbol{\sigma}$ and heat flux $\mathbf{Q} = -k\partial_{\mathbf{x}}T$

$$[\mathbf{v}] = \mathbf{0}, \quad \mathbf{n} \cdot [\boldsymbol{\sigma} + \boldsymbol{\tau}] = \mathbf{0}, \quad \mathbf{n} \cdot \left[(\boldsymbol{\sigma} + \gamma \mathbf{B}\mathbf{B}) \cdot \mathbf{v} + \tfrac{1}{2}\left(\varepsilon_0 |\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2\right)\mathbf{v} - \varepsilon_0 \mathbf{E} \times \mathbf{B} + k\partial_{\mathbf{x}}T\right] = 0, \tag{3.30}$$

where $\boldsymbol{\tau}$ is the electro-dynamic stress tensor defined by

$$\boldsymbol{\tau} = \tfrac{1}{2}(\varepsilon_0 |\mathbf{E}|^2 + (\mu_0^{-1} - 2\gamma)|\mathbf{B}|^2)\mathbf{I} - (\varepsilon_0 \mathbf{E}\mathbf{E} + (\mu_0 \mu)^{-1}\mathbf{B}\mathbf{B}).$$

Here $\mathbf{I}$ is the second-order identity tensor, and $\mathbf{n}$ denotes the unit normal vector to the material surface in the current configuration. The quantities enclosed by square brackets can in general be subjected to a discontinuity at the boundary surface of a moving material volume.

The initial conditions are

$$\mathbf{E}_{|t=0} = \mathbf{E}^0, \quad \mathbf{B}_{|t=0} = \mathbf{B}^0,$$

$$\rho_{|t=0} = \rho^0, \quad \mathbf{v}_{|t=0} = \mathbf{v}^0, \quad \boldsymbol{\sigma}_{|t=0} = \boldsymbol{\sigma}^0, \quad -k\partial_{\mathbf{x}} T_{|t=0} = \mathbf{Q}^0, \quad T_{|t=0} = T^0, \tag{3.31}$$

where functions with upper zero index are known.

It is easily seen that the system of constitutive equations (3.21)–(3.27) is fully interconnected with the complex system of boundary conditions Eq. (3.29) and (3.30).

4. Constitutive relations for hyperelastic MS materials

Constitutive equations for hyperelastic materials, frequently used to describe the non-linear elastic behavior of filled and unfilled elastomers, are obtained as the derivative of a strain-energy or stored-energy function Φ (see for example Green and Zerna, 1975; Holzapfel, 2001; Lurie, 1990; Ogden, 1997; Treloar, 1975). This scalar energy function is defined per unit reference volume rather than per unit mass. From there, the Cauchy stress tensor has the well known form

$$\boldsymbol{\sigma} = J^{-1} \frac{\partial \Phi(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}, \tag{4.1}$$

where the scalar $J = \det \mathbf{F} > 0$ is the volume ratio.

From Eqs. $(3.4)_2$ and (3.25) it follows that the strain-energy function is the Helmholtz free-energy function since $\Phi = \rho^0 \Psi$. However, the deformation tensor $\mathbf{F}$ is non-objective, see e.g. Lurie (1990), and Truesdell and Noll (1992), and thus, for mathematical modeling of hyperelastic MS materials we assume that $\Phi = \Phi(T, \mathbf{b}, \mathbf{B})$. In this equation $\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green or the Finger objective deformation tensor, $\mathbf{B}$ is the magnetic flux density and T the temperature. The Cauchy stress tensor is now obtained from

$$\boldsymbol{\sigma} = 2J^{-1} \frac{\partial \Phi(\mathbf{b})}{\partial \mathbf{b}} \cdot \mathbf{b} = 2J^{-1} \mathbf{b} \cdot \frac{\partial \Phi(\mathbf{b})}{\partial \mathbf{b}}. \tag{4.2}$$

Note, that for notational convenience, we do not distinguish between different strain-energy functions.

The Finger tensor $\mathbf{b}$ satisfies the well known Cayley–Hamilton identity

$$\mathbf{b}^3 - I_1 \mathbf{b}^2 + I_2 \mathbf{b} - I_3 \mathbf{I} = \mathbf{O}, \tag{4.3}$$

where $\mathbf{O}$ is the zero second-order tensor, $\mathbf{b}^k = \mathbf{b}^{k-1} \cdot \mathbf{b}$ $(k = 1, 2, 3)$ and $\mathbf{b}^0 = \mathbf{I}$. The strain invariants I_i are given by

$$\begin{aligned} I_1(\mathbf{b}) &= \operatorname{tr} \mathbf{b}, \\ I_2(\mathbf{b}) &= \tfrac{1}{2}[(\operatorname{tr} \mathbf{b})^2 - \operatorname{tr}(\mathbf{b}^2)], \\ I_3(\mathbf{b}) &= \det \mathbf{b} = J^2. \end{aligned} \tag{4.4}$$

It can be shown that the strain-energy function depends, for isotropic hyperelastic materials, on the three strain invariants only, (see for example Holzapfel, 2001; Lurie, 1990; Ogden, 2001). For the stress-free initial configuration where $I_1 = 3$, $I_2 = 3$ and $I_3 = 1$, the following relation, as shown by Drozdov (1996) and Lurie (1990), holds

$$\left(\frac{\partial \Phi}{\partial I_1} + 2\frac{\partial \Phi}{\partial I_2} + \frac{\partial \Phi}{\partial I_3}\right) = 0. \tag{4.5}$$

Constitutive material models to be used in finite elasticity are bound to satisfy some stringent mathematical requirements to allow for a physical consistent solution for all possible deformations. The most basic requirement implies that the value of the strain energy grows toward infinity for a continuum body expanding to infinity or for the body to compress to a point with vanishing volume, i.e. $\Phi \to +\infty$ as $\det\mathbf{b} \to +\infty$ or $\det\mathbf{b} \to +0$, (see e.g. Holzapfel, 2001; Lurie, 1990; Ogden, 1997). As a result, the strain-energy function is non-convex in $\mathbf{b}$, see Drozdov (1996).

The second requirement is uniqueness of a solution in finite elasticity boundary value problems as pointed out for example by Ciarlet (1988), Drozdov (1996) and Lurie (1990). Therefore, for the strain-energy function Φ the condition of *polyconvexity* in $\mathbf{b}$ was introduced by Ball (1977). This condition does not contradict basic hypotheses in continuum mechanics, but it is *sufficient* to establish the existence theorem for the global minimizer of the appropriate energy functional in finite elastostatics. At present the condition of polyconvexity of strain-energy functions is postulated as an additional restriction in hyperelasticity.

Ball (1977) showed that for an isotropic strain-energy function Φ, polyconvexity in $\mathbf{b}$ is equivalent to convexity in the strain invariants I_1, I_2 and I_3. We recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ ($n \geqslant 1$) is called *convex* if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and for any $t \in (0,1)$ the following inequality is satisfied:

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leqslant tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2).$$

As a result, the isotropic hyperelastic material may be described by a strain-energy function $\Phi = \Phi(I_1, I_2, I_3)$, which is convex in the invariants given by Eq. (4.4). The Cauchy stress tensor is then expressed in the Finger form, as shown by Drozdov (1996), Holzapfel (2001) or Lurie (1990)

$$\boldsymbol{\sigma} = 2J^{-1}\left[I_3\frac{\partial \Phi}{\partial I_3}\mathbf{I} + \left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\mathbf{b} - \frac{\partial \Phi}{\partial I_2}\mathbf{b}^2\right]. \tag{4.6}$$

For isotropic hyperelastic incompressible materials a suitable strain-energy function is given by

$$\Phi = \Phi(I_1, I_2) - \tfrac{1}{2}p(I_3 - 1), \tag{4.7}$$

where $p/2$ serves as an indeterminate *Lagrange multiplier*. In this case the Cauchy stress tensor has the modified Finger form

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\mathbf{b} - 2\frac{\partial \Phi}{\partial I_2}\mathbf{b}^2. \tag{4.8}$$

From the physical point of view, p is the hydrostatic pressure which is a supplementary thermodynamical variable. In this case the system of mechanical constitutive equations (3.25) and (3.26) is complemented by the incompressibility condition $J = 1$, as shown for example by Lurie (1990) or Ogden (2001).

From the theory of invariant, see Spencer (1971), it follows that an isotropic hyperelastic MS material can be fully described by the set of invariants

$$T, \quad |\mathbf{B}|^2, \quad I_1(\mathbf{b}), \quad I_2(\mathbf{b}), \quad I_3(\mathbf{b}),$$

and a set of pseudo-invariants

$$L_k = \mathbf{B}\cdot\mathbf{b}^k\cdot\mathbf{B}, \quad k = 1, 2, \ldots \tag{4.9}$$

But from the Cayley–Hamilton identity shown in Eq. (4.3), it follows that for an exponent $k \geqslant 3$, the corresponding pseudo-invariants L_k depend only on the invariants $|\mathbf{B}|^2$, I_1, I_2, I_3 and on the pseudo-invariants L_1 and L_2.

Thus, in the most general case the energy function Φ for isotropic hyperelastic MS material is given by

$$\Phi = \Phi(T, |\mathbf{B}|^2, I_1, I_2, I_3, L_1, L_2), \tag{4.10}$$

which is convex in the invariants I_1, I_2, I_3 and in the pseudo-invariants L_1, L_2.

The derivative of the pseudo-invariants with respect to the Finger tensor are given by

$$\frac{\partial L_1}{\partial \mathbf{b}} = \mathbf{BB}, \quad \frac{\partial L_2}{\partial \mathbf{b}} = \mathbf{b}\cdot\mathbf{BB} + \mathbf{BB}\cdot\mathbf{b},$$

and from Eqs. (4.2) and (4.6), we receive the augmented expression for the Cauchy stress tensor

$$\boldsymbol{\sigma} = 2J^{-1}\left[I_3\frac{\partial \Phi}{\partial I_3}\mathbf{I} + \left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\mathbf{b} - \frac{\partial \Phi}{\partial I_2}\mathbf{b}^2\right] + 2J^{-1}\left[\frac{\partial \Phi}{\partial L_1}\mathbf{BB}\cdot\mathbf{b} + \frac{\partial \Phi}{\partial L_2}(\mathbf{b}\cdot\mathbf{BB}\cdot\mathbf{b} + \mathbf{BB}\cdot\mathbf{b}^2)\right].$$

We known from the preceding section that for MS elastomers the Cauchy stress tensor is *symmetric*, which requires that $\partial\Phi/\partial L_1$ and $\partial\Phi/\partial L_2$ must vanish, i.e. Φ does not depend on the pseudo-invariant L_1 and L_2. From the physical point of view this result indicates that the magnetic flux density $\mathbf{B}$ *is not distorted as it passes through an isotropic medium*.

As a result, the strain-energy function of the isotropic MS elastomer depends only on the temperature T and on the intensity of the magnetic field $|\mathbf{B}|^2$.

To describe the mechanical behavior of an isotropic incompressible MS elastomer we propose, for simplicity, the following two parameter strain-energy function, which is polyconvex in $\mathbf{b}$:

$$\Phi = Gq^{-1}(I_1(\mathbf{b}) - 3)^{q/2}. \tag{4.11}$$

Here $G > 0$ is the shear modulus in the reference configuration, known from the linear theory, and $q > 1$ is the parameter of growth, which are functions of T and $|\mathbf{B}|^2$ only. For $q = 2$ we obtain the classical neo-Hookean model, which may be derived using concepts from the statistical theory of the elasticity of the molecular network structure of vulcanized rubber, see Treloar (1975).

We assume that the shear modulus G and the growth parameter q can be written in the form

$$G = G_0 + G_1|\mathbf{B}|^2, \quad q = q_0 + q_1|\mathbf{B}|^2, \tag{4.12}$$

where G_0, G_1, q_0 and q_1 are all positive; G_0 and q_0 are the field independent shear modulus and growth parameter. As a result, combining Eqs. (4.11) and (4.12), we obtain a four-parameter model for an isotropic incompressible MS elastomer, where G_0, G_1, q_0 and q_1 are functions of T only.

For a compressible MS elastomer the following strain-energy function can be used:

$$\Phi = Gq^{-1}(I_1(\mathbf{b}) - 3)^{q/2} + Kh(J), \tag{4.13}$$

where K is the bulk modulus in the reference configuration and is a function of T and $|\mathbf{B}|^2$ only. Here h: $(0, +\infty) \to [0, +\infty)$ is the function of compressibility such that $h(1) = 0$ and $h(J) \to \infty$ as $J \to +0$ or $J \to \infty$. For example, a good correlation of theoretical results and experiment data for compressible rubber-like materials is given by the function introduced by Ogden (1972)

$$h(J) = \beta^{-2}(\beta \log J + J^{-\beta} - 1), \tag{4.14}$$

where $\beta > 0$ is an empirical coefficient.

We assume again that

$$K = K_0 + K_1|\mathbf{B}|^2, \quad \beta = \beta_0 + \beta_1|\mathbf{B}|^2, \tag{4.15}$$

where K_0 and β_0 are both positive and represent the magnetic field independent bulk modulus and field independent volumetric parameter of growth, K_1 and β_1 are both positive. From Eqs. (4.12)–(4.15), we

obtain an eight-parameter model for an isotropic compressible MS elastomer, where G_0, G_1, q_0, q_1, K_0, K_1, β_0 and β_1 are functions of T only.

Remark 4.1. For the parameter $q = 1$ in Eq. (4.11) the existence of the limit static load and a discontinuous mapping with shear jumps of the first type have been shown, see Brigadnov (1993, 1996, 1999), It is suggested that from a mathematical point of view the solution of the elastostatic variational problem requires a relaxation process. The original partial relaxation is based on a discontinuous finite-element approximation, see Brigadnov (2001a,b). However, we will not discuss mathematical correctness of boundary value problems for MR elastomers in this paper. We plan to study this problem and numerical solution in an upcoming publication.

Remark 4.2. The dependence of the parameters G, q, K and β on $|\mathbf{B}|^2$ can be described by alternative theoretical or experimental functions as well, see for example Jolly et al. (1996a,b), Ginder and Davis (1994) and Rigbi and Jilken (1983).

5. Example: shear deformation between parallel plates

Let us consider the problem of a MS elastomer confined by two infinite parallel plates in the xy-plane and subjected to an unidirectional quasi-static shear deformation along the x-direction, see Fig. 1. Suppose that the mapping associated with the elastomer and that the magnetic field intensity perpendicular to the xy-plane, are given by

$$\mathbf{x} = \mathbf{X} + u(z)\mathbf{i}, \quad \mathbf{H} = H\mathbf{k}, \quad z \in [0, h], \tag{5.1}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the Cartesian basis vectors, u is the displacement along the x-direction, and h is the distance between the plates.

We shall assume that:

(1) the deformation is isothermal, i.e. $T = \text{const}$, $\partial/\partial T = 0$ and $R \equiv 0$;
(2) the hyperelastic MS elastomer is homogeneous, isotropic and incompressible;
(3) in the initial state the hydrostatic pressure vanishes;
(4) for the unidirectional shear, all values are constant in the y-direction, i.e. $\partial/\partial y = 0$;
(5) for the quasi-static deformation the inertia term in the left-hand side of Eq. (3.26) as well as initial conditions Eq. (3.31) are neglected;
(6) the body force is absent, i.e. $\mathbf{f} \equiv 0$;

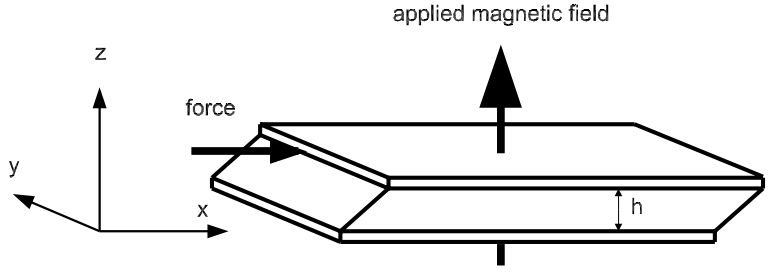

Fig. 1. Unidirectional shear of an incompressible MS elastomer between two infinite parallel plates subjected to a magnetic field perpendicular to shear direction.

(7) the shear deformation is due to a constant tangent force P_x, defined per unit length and per unit width along the x- and y-direction respectively. The force P_x is applied to the top plate along the x-direction;
(8) the elastomer adhere to the surfaces of the two plates;
(9) the magnetic field intensity is constant, i.e. $H = H_0 = \text{const}$ and $\mathbf{B} = B_0\mathbf{k}$ with $B_0 = \mu\mu_0 H_0$.

Within the framework of our assumptions, the constitutive equations (3.21)–(3.24) are satisfied for $\mathbf{E} \equiv \mathbf{0}$ and the power balance equation (3.27) is fulfilled automatically.

The Finger tensor has the form

$$\mathbf{b} = \mathbf{I} + u'^2\mathbf{ii} + u'(\mathbf{ik} + \mathbf{ki}),$$

and the condition of incompressibility $J = 1$ is fulfilled automatically.

We shall assume that the MS elastomer is described by the strain-energy function Eq. (4.11) and (4.12) with the positive constants G_0, G_1, q_0 and q_1.

From the general equation of the Cauchy stress tensor in Eq. (4.8), valid for incompressible elastomers, it now follows that the Cauchy stress tensor is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + G|u'|^{q-2}[\mathbf{I} + u'^2\mathbf{ii} + u'(\mathbf{ik} + \mathbf{ki})],$$

where p is the hydrostatic pressure. From this relation we find the stress components

$$\sigma_{xx} = -p + G|u'|^{q-2}(1 + u'^2),$$

$$\sigma_{yy} = \sigma_{zz} = -p + G|u'|^{q-2},$$

$$\sigma_{xz} = G|u'|^{q-1}, \quad \sigma_{xy} = \sigma_{yz} = 0.$$

From the equilibrium equation (3.26) and from our assumptions we obtain the system of differential equations

$$\begin{cases} \sigma'_{xz} = G((u')^{q-1})' = 0, \\ (-p + \sigma_{zz})' = 0, \end{cases} \tag{5.2}$$

with the boundary conditions

$$\begin{cases} \sigma_{xz}(h) = P_x, \\ u(0) = 0. \end{cases}$$

The solution of this boundary value problem has the simple form

$$u(z) = z\left(\frac{P_x}{G}\right)^{1/(q-1)}, \quad \sigma_{xz} = P_x, \quad \sigma_{xy} = \sigma_{yz} = 0, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = p = G\left(\frac{P_x}{G}\right)^{(q-2)/(q-1)}. \tag{5.3}$$

In the above numerical example we have used data from a commercially available MS elastomer, see Jolly et al. (1996a). The carrier elastomer was a mould using silicone oil loaded with a specified volume percent of carbonyl iron particles of a 3–4 μm mean diameter. The MS elastomer containing 30% iron by volume had the nominal shear modulus $G_0 \approx 1.8$ MPa, the yield magnetic flux density $B_y \approx 0.9$ T and the effective relative magnetic permeability $\mu \approx 7$, see Simon et al. (2001).

For $P_x = G_0$, the normalized displacement of the top plate along the x-direction is only a function of the constant magnetic flux density B_0:

$$v = \frac{u(h)}{h} = \varphi(B_0) = (1 + \eta B_0^2)^{1/(1-q_0-q_1B_0^2)}, \tag{5.4}$$

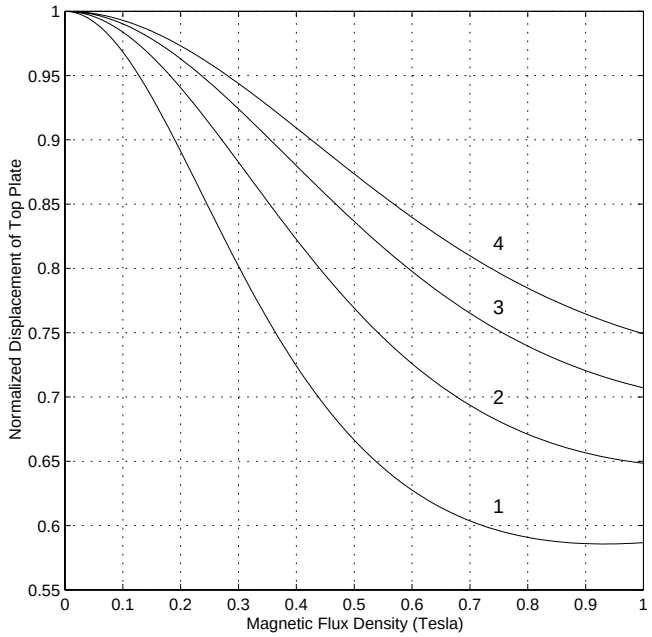

Fig. 2. The normalized displacement of the top plate as a function of the magnetic flux density. The curves 1, 2, 3 and 4 correspond, respectively, to the initial stretching parameter $q_0 = 1.3$, 1.6, 2.0 and 2.4.

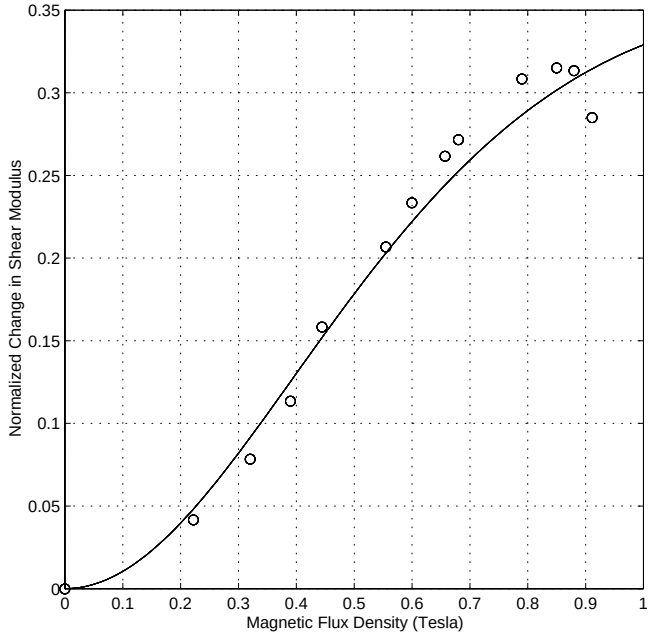

Fig. 3. The response of the shear modulus to an applied magnetic field for 30% iron by volume elastomer. Theoretical model data (—) using parameters $q_0 = 1.85$, $q_1 = 0.77$, $\eta = 0.91$ and experimental data (○) by Jolly et al. (1996a, Fig. 7).

where $\eta = G_1/G_0 > 0$. The normalized change in the shear modulus is given by the equation

$$\delta G = \frac{\Delta G}{G_0} = \psi(B_0) = 1 - \varphi(B_0), \tag{5.5}$$

where ΔG is the absolute change in the shear modulus, which was measured in physical experiments, see Jolly et al. (1996a,b).

In Fig. 2 the graphs of the Eq. (5.4) are shown for parameters $\eta = 1$ and $q_1 = 1$. The curves 1, 2, 3 and 4 correspond, respectively, to an initial growth parameter of $q_0 = 1.3$, 1.6, 2.0 and 2.4. It is easily seen that in the *pre-yield* region, i.e. for $B_0 \leqslant B_y$, increasing the magnetic field B_0 decreases the MS elastomer displacement.

In Fig. 3 the normalized change of the shear modulus for values of the parameters $q_0 = 1.85$, $q_1 = 0.77$ and $\eta = 0.91$ is shown. The values of these parameters were obtained by the method of minimal squares. Eq. (5.5) provides an acceptable agreement in the pre-yield region between results of the numerical simulation and the experimental observation. In reality, the presented model could be fully verified using experimental data which are, unfortunately, absent in open literature.

6. Conclusions

In this paper, we have summarized the complete system of constitutive equations for an isotropic MS elastomer within the framework of the electro-dynamical and thermo-mechanical theories. For hyperelastic, isotropic MS elastomers a simple strain-energy function has been presented and verified by experimental data. An acceptable agreement was illustrated between results of the numerical simulation and experimental observation. It was shown that the effect of the magnetic field is to stiffen the shear response of the material.

This mathematical framework can be used to develop more complex material laws of MS elastomers. However, the choice of the most appropriate model must be verified by detailed experimental data.

In the future we intend to use the presented simple strain-energy function to address issues related to the mathematical correctness and numerical solution of simple and complex boundary value problems.

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